Crash Course in Linear Algebra

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These class notes correspond to a 6 hours course on Introduction to Linear Algebra presented to the *CENTURI Master of Computational and Mathematical Biology* (CMB), Sept/2024, Marseille, France. This text is directed to biologists. Comments and suggestions can be sent to my e-mail, and they are very welcome.

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1 Introduction

Linear algebra is about transforming mathematical objects (named vectors) satisfying a *linearity property.* Let us see what it means: suppose that a *vector* $\mathbf{v}_1$ is transformed into a vector $\mathbf{u}_1$ by a transformation T, i.e.

$$T(\mathbf{v}_1) = \mathbf{u}_1,$$

and that a vector $\mathbf{v}_2$ is transformed into a vector $\mathbf{u}_2$ by the same transformation T,

$$T(\mathbf{v}_2) = \mathbf{u}_2.$$

Given any real number a (called a *scalar*), T is called a *Linear Transformation* if it satisfies the properties

$$T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2) = \mathbf{u}_1 + \mathbf{u}_2,$$

and

$$T(a\mathbf{v}_1) = aT(\mathbf{v}_1) = a\mathbf{u}_1.$$

In other words, T is a linear transformation if the "transformation of the sum" is the "sum of the transformations" and if transforming a "stretched" vector is the same as "stretching" the transformation of a vector.

Although it may seem very abstract, these properties are very important. If you are familiarized with physics, think about the gravitational force acting over an object: if you double the mass of this object, you double the force; If the object is a human holding a cat, the total force on the system is the force on the human *plus* the force on the cat. This happens because the gravitational force is *linear on the mass* (a similar example can be imagined for the Coulomb force acting on electrical charges). If you prefer to find a biological example, you can think about the solar energy absorbed by a leaf: increase the area of the leaf, and it will proportionally increase the energy it absorbs; if you increase the number of leaves, then the total absorbed energy is also going to increase proportionally.

Exercise: Try yourself to list other processes that show this property. Notice that in many cases such proportional "responses" can happen only if the "stimuli" are not so intense.

In the linear algebra field, all the concepts mentioned above (vectors, transformations, scalars, etc) are very well defined, and mathematicians study all the resulting properties of these objects and their applications. A basic tool in linear algebra is a *matrix.* They are used to represent the linear transformations and they have many features that can be used to understand general behaviours of a given system. As we will see, matrices emerge naturally in geometry and in equations describing time evolution, which are ubiquitous in biology, and that is why our focus in this "crash course" is going to be on matrix theory.

We shall start on basic operations and finish on Eigenvalue equations, with a later section on Dynamical systems as an important application.

2 Vectors and geometry

The object on which transformations act are *vectors*. Mathematically, vectors are well defined elements of a *vector space*, which is itself well-defined as a set equipped with 2 operations: *addition* and *multiplication by a scalar*. But apart from the formal definitions, the "traditional" vectors from physics and geometry are very good examples of such objects. So there is no problem if you keep them in mind.

Let us start with such vectors in two dimensions, i.e., on the Cartesian plane.

2.1 2D vectors

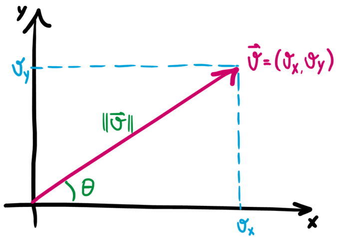

Figure 1: caption blablabla

A two dimensional vector $\mathbf{v}$ on the Cartesian plane is an ordered pair of real numbers (v_x, v_y). It is pictured as an arrow that starts at the origin of the Cartesian plane and ends at the point (v_x, v_y). Its *coordinates* are v_x in the x-direction (horizontal) and v_y in the y-direction (vertical). A vector can represent, for instance, the force acting over a particle. If such force pushes in a "diagonal" direction, so it pushes horizontally and vertically at the same time; the intensity of the force that is applied horizontally is given by $|v_x|$ and the intensity of the force that is applied vertically is given by $|v_y|$. The signals ($+$ or $-$) of the real numbers v_x and v_y give information about the direction *towards* which the force is pushing: left or right, up or down.

But you may imagine that when it is you who is pushing something, you do not think about making different strengths in different directions; you just choose an *angle* and apply a certain strength. Such "total" strength is the *magnitude* of the vector (its length, in geometry) and the direction it is applied is the angle θ between the vector and the x-axis. In linear algebra, the length of the vector is called its *norm*, represented with double vertical lines: $\|\mathbf{v}\|$. Calling $\|\mathbf{v}\| = r$, the pair (r, θ) defines the same vector $\mathbf{v}$, but now in *polar coordinates*.

We can translate the system from Cartesian to polar coordinates with some trigonometry,

From (v_x, v_y) to (r, θ)	From (r, θ) to (v_x, v_y)
$r = \sqrt{v_x^2 + v_y^2}$	$v_x = r\cos\theta$
$\tan\theta = v_x/v_y$	$v_y = r\sin\theta$

If a vector has all coordinates equal to zero, then it is called a null vector, and is represented by $\mathbf{0}$.

Suppose now that you and your friend are pushing a box; the resultant force is the sum of the force you apply and the force your friend applies. Therefore, vectors are being summed. In Cartesian coordinates, it is quite easy to sum vectors (we shall keep this *coordinate system* from now on). Given the vectors $\mathbf{v} = (v_x, v_y)$ and $\mathbf{u} = (u_x, u_y)$,

$$\mathbf{v} + \mathbf{u} = (v_x, v_y) + (u_x, u_y) = (v_x + u_x, v_y + u_y), \tag{1}$$

(you add the coordinates). Your friend may also ask you "push harder!" which is the same as multiplying the force you apply by a scalar a,

$$a\mathbf{v} = (av_x, av_y), \tag{2}$$

(you multiply both coordinates). If $a < 0$, then instead of pushing the box, you would be pulling it: you reverse the direction of the vector. Multiplying a vector by a scalar is the same as stretching (or compressing) it. A vector on the plane defines a line, and every point on this line can be reached if you stretch that initial vector enough. Hence, the multiplication by a scalar does not result in a vector that is outside of this line: a single vector does not *generate* the whole plane; you need at least two!

Notice that $\mathbf{v} + \mathbf{u}$ and $a\mathbf{v}$ are *also* vectors in 2 dimensions.

If two vectors *do not generate the same line*, then they can be combined to generate any other vector in two dimensions,

$$\mathbf{w} = a\mathbf{v} + b\mathbf{u}. \tag{3}$$

This is called a *linear combination* of the vectors $\mathbf{v}$ and $\mathbf{u}$. $\mathbf{w}$ can be any two dimensional vector if, and only if, $\mathbf{u}$ and $\mathbf{v}$ are *linearly independent*, i.e., one cannot generate $\mathbf{v}$ by stretching $\mathbf{u}$ and vice-versa. Mathematically, it translates to:

Pay attention to this condition! Think about a triangle: when cannot 3 points form a triangle? How does it translate to vectors? (Draw the points on a piece of paper and the vectors starting or ending on these points).

Two non-null vectors $\mathbf{u}$ and $\mathbf{v}$ are **linearly independent** (LI) if, and only if,

$$a\mathbf{v} + b\mathbf{u} = \mathbf{0}$$

only when $a = 0$ and $b = 0$.

Because of the way the Cartesian plane is constructed, if you have a set of 2 LI vectors, you cannot find a third vector that is simultaneously LI to the first two vectors. With a pair of LI vectors, all the plane can be generated (and this is one way to say that the Cartesian plane has *dimension* equal to 2). Such pair is called a *basis* of the plane. A common choice for a basis of the plane is the set of vectors $(1, 0)$ and $(0, 1)$. These vectors describe the x-axis and y-axis, respectively; the angle between them is $\pi/2$ and their length is one. These special properties make such basis very useful, which is the reason it receives a special name: the *canonical basis*, commonly represented as

Exercise: Show that if $\mathbf{v}$ and $\mathbf{u}$ are LI vectors in the plane, a third vector $\mathbf{w}$ also in the plane cannot be simultaneously independent to $\mathbf{v}$ and $\mathbf{u}$.

$$\hat{\imath} = (1, 0) \text{ and } \hat{\jmath} = (1, 0).$$

The notation with a hat usually refers to vectors that have size 1, and they are commonly called *versors*.

Exercise: Show that $\hat{\imath}$ and $\hat{\jmath}$ are linearly independent.

Exercise: Consider the vector $\mathbf{v} = 1/\sqrt{2}(1, 1)$. Find another vector LI to $\mathbf{v}$. Use a linear combination of such vectors to generate the canonical basis $\hat{\imath}$ and $\hat{\jmath}$.

With the canonical basis, we can write

$$\mathbf{v} = (v_x, v_y) = v_x(1, 0) + v_y(0, 1) = v_x\hat{\imath} + v_y\hat{\jmath},$$
$$\mathbf{u} = (u_x, u_y) = u_x(1, 0) + u_y(0, 1) = u_x\hat{\imath} + u_y\hat{\jmath},$$

and their sum is given by

$$\mathbf{v}+\mathbf{u} = (v_x+u_x, v_y+u_y) = (v_x+u_x)(1, 0)+(v_y+u_y)(0, 1) = (v_x+u_x)\hat{\imath}+(v_y+u_y)\hat{\jmath},$$

and the multiplication by a scalar is given by

$$a\mathbf{v} = av_x\hat{\imath} + av_y\hat{\jmath}.$$

Another notation which is going to be very useful is the column notation: instead of displaying the vectors as pairs (v_x, v_y), we can write them as columns $\mathbf{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$. For instance, the canonical basis is written as $\hat{\imath} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\hat{\jmath} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

All these notions can be easily extended to more than two dimensions. To represent a vector in 3 dimensions, we can include a third coordinate, $\mathbf{v} = (v_x, v_y, v_z)$ and the operations of sum and multiplication by scalar happen as before,

$$\begin{aligned} \mathbf{v} + \mathbf{u} &= (v_x, v_y, v_z) + (u_x, u_y, u_z) = (v_x + u_x +, v_y + u_y, v_z + u_z), \\ a\mathbf{v} &= (av_x, av_y, av_z), \end{aligned}$$

and in this case, at least three vectors (all LI from one another) are needed to describe the whole three dimensional space.

Exercise: What is the canonical basis of the three-dimensional real space? And how would it be for the n-dimensional real space?

3 Matrices

Using the canonical basis we can write any vector $\mathbf{v}$ as

$$\mathbf{v} = v_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}.$$

But we can also try to write the same vector $\mathbf{v}$ with the aid of a different basis, e.g. $\hat{I} = (I_x, I_y)$ and $\hat{J} = (J_x, J_y)$. In this new basis, $\mathbf{v}$ is given by a different linear combination of the vectors $\hat{I}$ and $\hat{J}$, with different coordinates a and b,

$$\mathbf{v} = a\hat{I} + b\hat{J} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}. \tag{4}$$

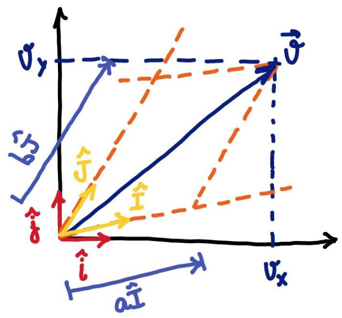

Figure 2: caption blablabla

In order to make this notation more compact, let us define the object

$$\mathbb{A} = \begin{bmatrix} \hat{I} & \hat{J} \end{bmatrix} = \begin{bmatrix} I_x & J_x \\ I_y & J_y \end{bmatrix},$$

in which we have put the basis vectors side by side. Now, by defining the operation

$$\begin{bmatrix} \hat{I} & \hat{J} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \end{bmatrix},$$

and comparing this equation with equation (4), we see that

$$\underbrace{\begin{bmatrix} \hat{I} & \hat{J} \end{bmatrix}}_{=\mathbb{A}} \begin{bmatrix} a \\ b \end{bmatrix} = a\hat{I} + b\hat{J}. \tag{5}$$

Therefore, the way that the object $\mathbb{A}$ acts on a pair of numbers, which is itself a vector, defines a new vector: it *transforms* the vector (a, b) into the vector (v_x, v_y). Now, let us see how the object $\mathbb{A}$ transforms a vector $\mathbf{w} = \mathbf{v} + \mathbf{u}$:

$$\mathbb{A}\mathbf{w} = \begin{bmatrix} \hat{I} & \hat{J} \end{bmatrix} \begin{bmatrix} v_x + u_x \\ v_y + u_y \end{bmatrix} = (v_x + u_x)\hat{I} + (v_y + u_y)\hat{J} = (v_x\hat{I} + v_y\hat{J}) + (u_x\hat{I} + u_y\hat{J}) = \begin{bmatrix} \hat{I} & \hat{J} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} + \begin{bmatrix} \hat{I} & \hat{J} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}.$$

Therefore,

$$\mathbb{A}(\mathbf{v} + \mathbf{u}) = \mathbb{A}\mathbf{v} + \mathbb{A}\mathbf{u}. \tag{6}$$

Also, if $\mathbf{w} = a\mathbf{v}$,

$$\mathbb{A}\mathbf{w} = \begin{bmatrix} \hat{I} & \hat{J} \end{bmatrix} \begin{bmatrix} av_x \\ av_y \end{bmatrix} = av_x\hat{I} + av_y\hat{J} = a(v_x\hat{I} + v_y\hat{J}) = a \begin{bmatrix} \hat{I} & \hat{J} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix}.$$

and then

$$\mathbb{A}(a\mathbf{v}) = a\mathbb{A}(\mathbf{v}). \tag{7}$$

Exercise: Show that for a linear transformation $\mathbb{A}$, it is valid that $\mathbb{A}(\mathbf{v} + k\mathbf{u}) = \mathbb{A}\mathbf{v} + k\mathbb{A}\mathbf{u}$, where k is a scalar.

We can see that the way $\mathbb{A}$ transforms a vector follows interesting (and important) properties, which are called *linear* properties and thus $\mathbb{A}$ defines a *linear transformation*. It is not hard to ague now that the object defined in $\mathbb{A}$ deserves a special name and a careful study, and this is what we are going to do now.

Exercise: Is the transformation $T(x, y) = (xy, y)$ linear? And what about $T(x, y) = (x - y, x + y)$?

3.1 Definitions

Without any surprise, the object $\mathbb{A}$ is called a *matrix*. We shall define it more generally now.

Given two integer numbers $m > 0$ and $n > 0$, a matrix $\mathbb{A}_{m\times n}$ is a collection of double-indexed elements a_{ij}, with $1 \leq i \leq m$ and $1 \leq j \leq n$, organized as a table with m rows and n columns:

$$\mathbb{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}. \tag{8}$$

Notations: In these notes, vectors are written with bold letters, $\mathbf{u}, \mathbf{v}, \ldots$, and their elements with small letters and a single index $u_i, v_i, \ldots$ For matrices, I am gonna use double-line letters, $\mathbb{A}, \mathbb{B}, \ldots$ and, sometimes, capital Greek letters, as Λ and Σ. The elements of a matrix are going to be their corresponding small letters followed by their double-indexes, $a_{ij}, b_{ij}, \ldots$. The brackets around a matrix and around a column vector are chosen to be of square type.

Some examples of real matrices are:

$$\mathbb{A} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 8 \\ -4 & 2 & -1 \end{bmatrix} \quad \mathbb{B} = \begin{bmatrix} 5 & -1 \\ 2 & 0 \\ 2 & 1 \end{bmatrix} \quad \mathbb{C} = \begin{bmatrix} 4 & 0 & 9 \\ 2 & -1 & 3 \end{bmatrix} \quad \mathbb{D} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

(i) $\mathbb{A}$ is a *square matrix*, i.e. it has the same number of rows and columns, $m = n = 3$ (it is a 3×3 matrix); n in this case is called the order of the matrix.

(ii) $\mathbb{B}$ and $\mathbb{C}$ are rectangular matrices. $\mathbb{B}$ has 3 rows and 2 columns, (it is a 3×2 matrix), and $\mathbb{C}$ has 2 rows and 3 columns, (it is a 2×3 matrix).

(iii) $\mathbb{D}$ is a 4×4 square matrix, but with the very special property that all its elements except the *main diagonal* are zero. The main diagonal of a square matrix $\mathbb{A}$ is the set of elements a_{ij} with $i = j$. A matrix like $\mathbb{D}$ is called *diagonal*.

3.2 Basic Operations

We start with stating that: two matrices $\mathbb{A}$ and $\mathbb{B}$ are equal if, and only if, their elements are equal, i.e.

Given two $n \times m$ matrices $\mathbb{A}$ and $\mathbb{B}$,

$$\mathbb{A} = \mathbb{B} \Leftrightarrow a_{ij} = b_{ij}, \text{ for every } 1 \leq i \leq n \text{ and } 1 \leq j \leq m. \tag{9}$$

3.2.1 Addition

Two matrices $\mathbb{A}$ and $\mathbb{B}$ with the same dimensions can be added. This operation is defined as summing element by element,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \ldots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \ldots & a_{mn}+b_{mn} \end{bmatrix},$$

or,

Given two $n \times m$ matrices $\mathbb{A}$ and $\mathbb{B}$, their sum $\mathbb{A} + \mathbb{B} = \mathbb{C}$ is also a $n \times m$ matrix, whose elements are given by

$$c_{ij} = (\mathbb{C})_{ij} = (\mathbb{A} + \mathbb{B})_{ij} = (\mathbb{A})_{ij} + (\mathbb{B})_{ij} = a_{ij} + b_{ij}. \tag{10}$$

Properties The addition of matrices is *commutative*, $(\mathbb{A}+\mathbb{B}=\mathbb{B}+\mathbb{A})$ and *associative*, $((\mathbb{A}+\mathbb{B})+\mathbb{C}=\mathbb{A}+(\mathbb{B}+\mathbb{C}))$, it has a *neutral element* $\mathbb{O}$, $(\mathbb{A}+\mathbb{O}=\mathbb{O}+\mathbb{A}=\mathbb{A})$ and every matrix $\mathbb{A}$ has an inverse element $-\mathbb{A}$, $(\mathbb{A}+(-\mathbb{A})=\mathbb{O})$.

Exercise: Given the definition of the addition operation, prove these properties, i.e., show it is commutative and associative and find the elements $\mathbb{O}$ and $-\mathbb{A}$.

3.2.2 Scalar multiplication

We are also able to multiply a matrix by a scalar. We must only multiply every element of the matrix by the same number. If this number is k,

$$k\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}$$

or,

Given a $n\times m$ matrix $\mathbb{A}$ and a real number k, the matrix $\mathbb{B}=k\mathbb{A}$ is also a $n\times m$ matrix, whose elements are given by

$$b_{ij}=(\mathbb{B})_{ij}=(k\mathbb{A})_{ij}=k(\mathbb{A})_{ij}=ka_{ij}. \tag{11}$$

Properties Given two scalars μ and λ, the scalar multiplication is *associative* $(\mu(\lambda\mathbb{A})=(\mu\lambda)\mathbb{A})$ and it has a *neutral element* p, $(p\mathbb{A}=\mathbb{A})$. Also, together with the addition operation, it is *distributive*, $(\mu(\mathbb{A}+\mathbb{B})=\mu\mathbb{A}+\mu\mathbb{B}$ and $(\mu+\lambda)\mathbb{A}=\mu\mathbb{A}+\lambda\mathbb{A})$.

Exercise: Given the definition of the scalar multiplication operation, prove these properties, i.e., show it is associative and distributive and find the scalar p.

3.2.3 The transpose

The *transposed matrix* $\mathbb{A}^T$ of a $n\times m$ matrix $\mathbb{A}$ is a $m\times n$ matrix whose elements are those of $\mathbb{A}$ but with inverted indexes, i.e.

$$(\mathbb{A}^T)_{ij}=a_{ji}. \tag{12}$$

Some properties of this operation are:

(i) The main diagonal of a square matrix is invariant under transposing a matrix,

$$(\mathbb{A}^T)_{ii}=a_{ii}. \tag{13}$$

(ii) The transpose of the transpose of $\mathbb{A}$ is equal to $\mathbb{A}$,

$$(\mathbb{A}^T)^T=\mathbb{A}. \tag{14}$$

(iii) The transpose of the sum of matrices is the sum of the transposes,

$$(\mathbb{A}+\mathbb{B})^T=\mathbb{A}^T+\mathbb{B}^T. \tag{15}$$

(iv) Something similar happens for the multiplication by a scalar,

$$(k\mathbb{A})^T=k\mathbb{A}^T. \tag{16}$$

Exercise: Prove the properties from (i) to (iv).

Symmetric Matrix: A square matrix $\mathbb{A}$ is called symmetric if, and only if, it satisfies

$$\mathbb{A}^T=\mathbb{A}. \tag{17}$$

3.2.4 Trace

The *trace* of a square matrix $\mathbb{A}$ of order n, (written as $\mathrm{tr}(\mathbb{A})$), is the sum of the elements of its main diagonal, i.e.

$$\mathrm{tr}(\mathbb{A}) = \sum_{i=1}^{n} a_{ii}. \tag{18}$$

The trace is a *linear map*, i.e., it connects a matrix with a real number and follows linear properties,

(i) Given two square matrices $\mathbb{A}$ and $\mathbb{B}$ of the same order,

$$\mathrm{tr}(\mathbb{A} + \mathbb{B}) = \mathrm{tr}(\mathbb{A}) + \mathrm{tr}(\mathbb{B}). \tag{19}$$

(ii) Given a square matrix $\mathbb{A}$ and a scalar k,

$$\mathrm{tr}(k\mathbb{A}) = k\mathrm{tr}(\mathbb{A}). \tag{20}$$

It is also not hard to see that, because the transposition operation does not change the main diagonal, $\mathrm{tr}(\mathbb{A}^T) = \mathrm{tr}(\mathbb{A})$.

4 Matrix multiplication

Remember that a matrix acts on a vector and transforms it into another vector, for instance

$$\mathbb{A}\mathbf{v} = \mathbf{u}.$$

We could then act on the resulting vector $\mathbf{u}$ with another matrix $\mathbb{B}$, as $\mathbb{B}\mathbf{u} = \mathbf{w}$ and then the vector $\mathbf{w}$ would be

$$\mathbb{B}(\mathbb{A}\mathbf{v}) = \mathbf{w}.$$

It would be very interesting if we could combine the sequential effects of $\mathbb{A}$ and $\mathbb{B}$ into a single matrix $(\mathbb{B}\mathbb{A})$ which has exactly the same effect on $\mathbf{v}$, transforming it into $\mathbf{w}$,

$$(\mathbb{B}\mathbb{A})\mathbf{v} = \mathbb{B}(\mathbb{A}\mathbf{v}) = \mathbf{w}.$$

It is about combining these effects that matrix multiplication is about. Equation (5) already gives us a hint of how this operation should be performed: "a row times a column".

4.1 Definition

Given a $n \times m$ matrix $\mathbb{A}$ and a $m \times p$ matrix $\mathbb{B}$, the *matrix multiplication* $\mathbb{A}\mathbb{B} = \mathbb{C}$ is a $n \times p$ matrix, whose elements are given by

$$c_{ij} = (\mathbb{C})_{ij} = (\mathbb{A}\mathbb{B})_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}. \tag{21}$$

Putting this expression into words:

- You can multiply two matrices if the number of columns of the first matrix equals the number of rows of the second matrix (it implicitly says that the order of the matrices matter!)

Exercise: Multiply a column matrix $\mathbb{A}_{2\times 1}$ by its transpose, $\mathbb{A}^T\mathbb{A}$ and $\mathbb{A}\mathbb{A}^T$. If $\mathbb{A}$ represents a vector, what does $\mathbb{A}^T\mathbb{A}$ represent?

- The resultant matrix is going to have the same number of rows as the first matrix and the same number of columns as the second matrix.
- To obtain the element ij of the resultant matrix, one should multiply the first element of the row i of matrix $\mathbb{A}$ by the first element of the column j of $\mathbb{B}$; multiply the second element of the row i of matrix $\mathbb{A}$ by the second element of the column j of $\mathbb{B}$; and successively up to the m-th element of the row i of $\mathbb{A}$ and the m-th element of the column j of $\mathbb{B}$. At the end, add all them up.

$$\mathbb{A}_{n\times m}\mathbb{B}_{m\times p} = \mathbb{C}_{n\times p}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mp} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{np} \end{bmatrix}$$

$$c_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} + \ldots + a_{2m}b_{m1}$$

Exercise: Multiply the matrices $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ \mu & 1 \end{bmatrix}$, where μ is a real number. Check that the resulting matrix is composed by two linearly independent vectors, regardless the value of μ.

4.1.1 Properties

(a) It is non-commutative: The order in which you multiply two different matrices matter, even if they are both square matrices:

$$\mathbb{A}\mathbb{B} \neq \mathbb{B}\mathbb{A}. \tag{22}$$

(b) It is associative: For three matrices $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$,

$$(\mathbb{A}\mathbb{B})\mathbb{C} = \mathbb{A}(\mathbb{B}\mathbb{C}). \tag{23}$$

(b) It is distributive: For three matrices $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$,

$$(\mathbb{A} + \mathbb{B})\mathbb{C} = \mathbb{A}\mathbb{B} + \mathbb{A}\mathbb{C}. \tag{24}$$

(c) There is a neutral element: If $\mathbb{A}$ is a square matrix of order n, then there exists an *identity matrix* $\mathbb{I}_n$ such that

$$\mathbb{I}_n\mathbb{A} = \mathbb{A}\mathbb{I}_n = \mathbb{A}. \tag{25}$$

Exercise: Show that the neutral element is unique, i.e., there is only one matrix of order n that satisfies the neutral element property. (**Hint:** Suppose there are different neutral elements and then show they are equal to each other).

Exercise: Show that the identity matrix as defined in Eq. (26) satisfies the neutral element property.

The matrix $\mathbb{I}_n$ has n rows and n columns, with zeros everywhere except for the main diagonal, which is filled with 1s,

$$\mathbb{I}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix},$$

or simply

$$(\mathbb{I}_n)_{ij} = \begin{cases} 1 \text{ , if } i = j; \\ 0 \text{ , otherwise.} \end{cases} \tag{26}$$

(d) Inverse matrix: A square matrix $\mathbb{A}$ of order n may have an *inverse* $\mathbb{A}^{-1}$, i.e.,

$$\mathbb{A}^{-1}\mathbb{A} = \mathbb{A}\mathbb{A}^{-1} = \mathbb{I}_n. \tag{27}$$

If $\mathbb{A}$ and $\mathbb{B}$ have an inverse, then

$$(\mathbb{A}\mathbb{B})^{-1} = \mathbb{B}^{-1}\mathbb{A}^{-1}. \tag{28}$$

Exercise: Show that the inverse of $\mathbb{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is $\mathbb{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$.

(e) Transpose of a product: Given two matrices $\mathbb{A}$ and $\mathbb{B}$,

$$(\mathbb{A}\mathbb{B})^T = \mathbb{B}^T\mathbb{A}^T. \tag{29}$$

(f) k-th power of a matrix: If $k = 1, 2, \ldots$ and $\mathbb{A}$ is a square matrix, then

$$\mathbb{A}^k = \underbrace{\mathbb{A}\mathbb{A}\ldots\mathbb{A}}_{k \text{ times}}. \tag{30}$$

Exercise: Show that if $\mathbb{A}$ is a diagonal matrix then

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{nn} \end{bmatrix}^k = \begin{bmatrix} a_{11}^k & 0 & \cdots & 0 \\ 0 & a_{22}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{nn}^k \end{bmatrix}$$

(g) Exponential of a matrix: Given $\mathbb{A}$ a square matrix of order n, we define

$$e^{\mathbb{A}} = \sum_{k=0}^{\infty} \frac{\mathbb{A}^k}{k!} = \mathbb{I}_n + \mathbb{A} + \frac{1}{2!}\mathbb{A}^2 + \frac{1}{3!}\mathbb{A}^3 + \ldots \tag{31}$$

where $\mathbb{A}^0 \equiv \mathbb{I}_n$. This formal definition of the exponential of a matrix is useful in the study of dynamical systems, as we are going to have as an application at the end of this course.

4.2 Matrices as transformations

Now that we know how to multiply matrices, we can apply them on vectors. We already introduced the notation of a column vector and the reason behind it is because it works exactly as a matrix of n rows and 1 column, with n being the dimension of the vector (for vectors in the plane, $n = 2$). Thus in order to apply a matrix (i.e. the linear transformation) on a vector, we multiply them as a matrix multiplication,

$$\mathbb{A}\mathbf{v} = \mathbf{u} \iff \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \ldots + a_{mn}v_n \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

As the simplest example, consider the identity matrix $\mathbb{I}_n$: its action on a vector is to keep it as it is,

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

Consider now the 2×2 matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. What is the effect of this transformation? Notice that applying it on a vector $\begin{bmatrix} v_x \\ v_y \end{bmatrix}$ inverts the signal of the

y-coordinate $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}v_x\\v_y\end{bmatrix} = \begin{bmatrix}v_x\\-v_y\end{bmatrix}$. Thus this transformation is a *reflection* around the x-axis.

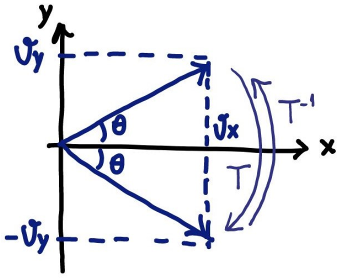

Figure 3: caption blablabla

Now, the inverse of a matrix can be understood as the "reverse" transformation. For the previous example, to reverse a reflection you simply need to reflect the vector again! Thus the inverse of the matrix $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ is exactly $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$.

Now we can also understand why matrix multiplication is non-commutative: because the order of the transformation matters. For instance, a rotation around an axis is a valid transformation (and it is as important as it deserves its own section). If we call the rotations of $\pi/2$ around the axis x, y and z as $\mathbb{R}_x$, $\mathbb{R}_y$ and $\mathbb{R}_z$, respectively, then the sequence of rotations $\mathbb{R}_z\mathbb{R}_y\mathbb{R}_x$ is not equivalent to $\mathbb{R}_x\mathbb{R}_z\mathbb{R}_y$. As Figure 4 shows, you would have a hard time if you choose the wrong sequence to try to lay down.

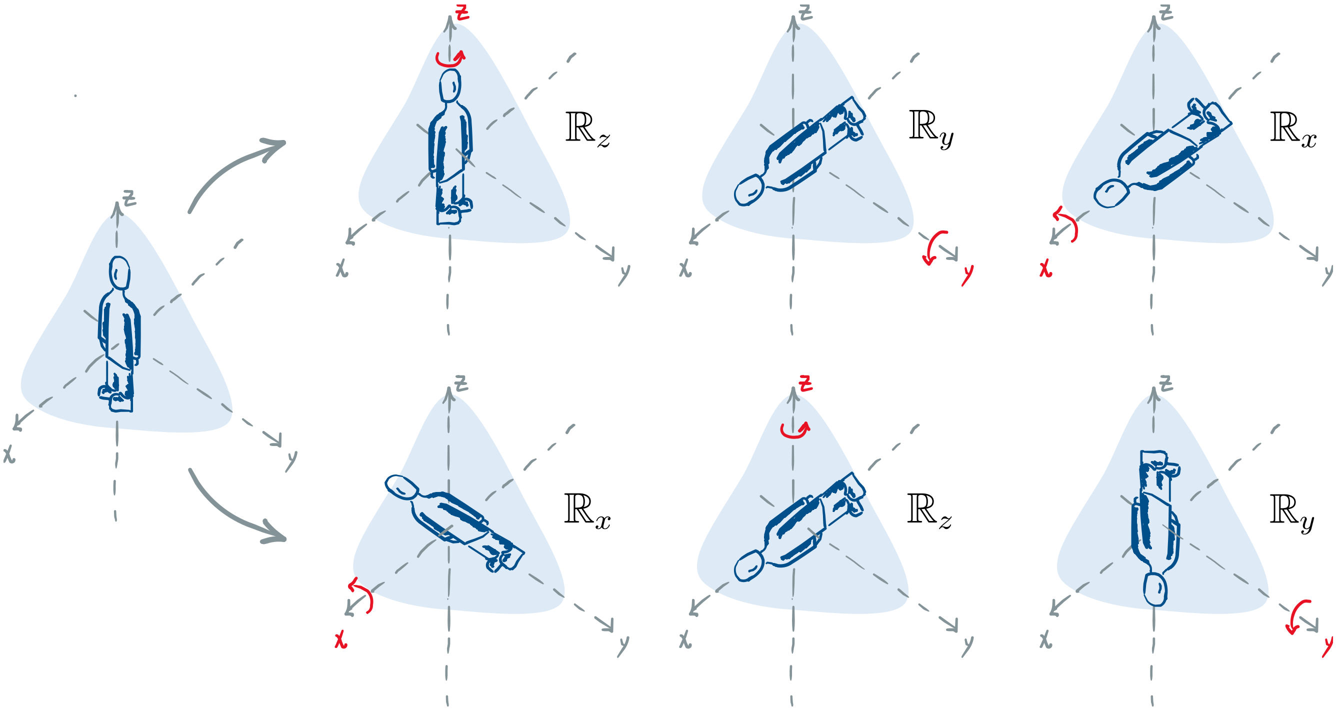

Figure 4: Caption

4.3 Rotation Matrix in 2D

Rotating vectors appear many times in physics. It is an essential part of the motion we observe in nature and in the universe: planets spin around their axis and rotate around their orbits; bugs fly in circles around lamps; rotating wheels allow cars to go everywhere; kids play with yoyo toys; clocks move their hands... But rotation is more than that: it is the prototype of a cyclical motion and cycles are ubiquitous in nature: the cyrcadian rhythm, the beat of cardiac cells, the membrane potential of neurons, the respiration cycles of eukaryotic cells, the peristaltic movements of the digestive system... *Oscillatory motions* are everywhere.

We shall now study how a rotation in the plane looks like. Given a vector $\mathbf{v} = \begin{bmatrix}v_x\\v_y\end{bmatrix}$, of length $r = \sqrt{v_x^2 + v_y^2}$ and angle $\alpha = \tan^{-1}(v_x/v_y)$, how would it look like if it is rotated around the origin by an angle θ? Since the length does not change (i.e., rotations preserve the norm of a vector), the only effect it has is to increase the angular direction α by an amount θ. Therefore, the new vector $\mathbf{u}$, after a rotation, is given by $\begin{bmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{bmatrix}$.

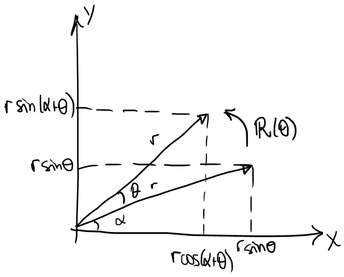

Figure 5: caption blablabla

Let us call the rotation by an amount θ the object $\mathcal{R}_\theta$. Using the vectors defined above, we may write $\mathcal{R}_\theta \mathbf{v} = \mathbf{u}$. If c is a scalar and $\mathbf{w}$ is another vector, it is possible to show that

$$\mathcal{R}_\theta(\mathbf{v}+\mathbf{w}) = \mathcal{R}_\theta \mathbf{v} + \mathcal{R}_\theta \mathbf{w}$$
$$\mathcal{R}_\theta(c\mathbf{v}) = c\mathcal{R}_\theta \mathbf{v}$$

and thus the transformation $\mathcal{R}(\theta)$ is linear. Therefore, it has a matrix representation, which we call $\mathbb{R}(\theta)$, which directly acts on 2D vectors. Now,

$$\mathbf{u} = \begin{bmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{bmatrix} = \begin{bmatrix} r\cos\alpha\cos\theta - r\sin\alpha\sin\theta \\ r\sin\alpha\cos\theta + r\cos\alpha\sin\theta \end{bmatrix}.$$

But $v_x = r\cos\alpha$ and $v_y = r\sin\alpha$, so

$$\mathbf{u} = \begin{bmatrix} v_x\cos\theta - v_y\sin\theta \\ v_y\cos\theta + v_x\sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \mathbb{R}(\theta)\mathbf{v},$$

where we have recognized the *rotation matrix* $\mathbb{R}(\theta)$.

The *rotation matrix* $\mathbb{R}(\theta)$ of an angle θ of 2D vectors (in the canonical basis) is equal to

$$\mathbb{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{32}$$

Rotations are reversible operations: to go back from a rotation of angle θ, you just need to rotate it of an angle $-\theta$. Thus,

$$\mathbb{R}(\theta)\mathbb{R}(-\theta) = \mathbb{I}_2.$$

Hence, the inverse matrix of a rotation matrix is given by

$$\mathbb{R}(\theta)^{-1} = \mathbb{R}(-\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{33}$$

Exercise: With matrix multiplication, show that this is correct.

Another important property is that, unlike rotations in 3 dimensions, rotations in the plane are commutative, i.e., rotating by an angle θ and then by angle α is the same as rotating first by an angle α and then by an angle θ. Moreover, it is the same as the rotation of an angle $\theta+\alpha$,

$$\mathbb{R}(\alpha)\mathbb{R}(\theta) = \mathbb{R}(\theta)\mathbb{R}(\alpha) = \mathbb{R}(\theta+\alpha). \tag{34}$$

Exercise: Prove that the rotation matrix satisfies these identities.

5 Determinants

We already mentioned a quantity that we can extract from a matrix with interesting properties, called the trace of a matrix. We are now going to introduce a different quantity called the *determinant* of a matrix, which has different uses in different contexts. It is calculated for square matrices and it can provide information about "repeated lines" (which is something quite useful in linear algebra).

Given a square matrix $\mathbb{A}$ of order n, choose n distinct elements of $\mathbb{A}$ without repeating any row or column. Multiply all of these numbers. Now, choose a different set of n elements satisfying the same rule and multiply them all altogether. Repeat this process to all of the different possible sets. There are $n!$ different ways of doing it. Now, there is a signal rule which we are not going to discuss here, but half of these products you should multiply by -1 and the other half by $+1$. Sum all the $n!$ results *et voilà*, you have calculated the determinant of $\mathbb{A}$.

I know it does not seem an easy thing, and it may not be for large matrices, but the determinant of a matrix still bears many good properties. Let us explicitly calculate the 2D case and the 3D case and we then describe its properties.

5.1 2 × 2 Matrix

The prescription presented before is easy for a 2×2 matrix. There are only 2 different sets of numbers that satisfy the rule "different rows and different columns",

$$\{a_{11}, a_{22}\} \text{ and } \{a_{12}, a_{21}\}.$$

Following the prescription, one of the sets is multiplied by $+1$ (the first set – Again, I am not discussing the reason here) and the other set is going to be multiplied by -1 (the second set). The determinant is then calculated as

$$\det(\mathbb{A}) = a_{11}a_{22} - a_{12}a_{21}. \tag{35}$$

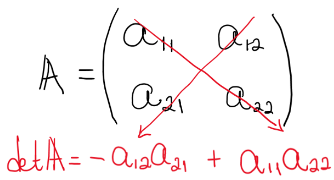

Figure 6: caption blablabla

Notation: There are two main notations for the determinant of a matrix: $\det(\mathbb{A})$ and $|\mathbb{A}|$. In these notes, I will prefer $\det(\mathbb{A})$ over the $|\mathbb{A}|$, but the last may also appear. So do not get scared, they mean the same thing!

Fig. 6 shows the graphical strategy to calculate such value: you calculate the products in the different diagonals. When the diagonal goes from left to right, it is positive; when it comes from right to left, it is negative.

For instance, the determinant of the rotation matrix is given by

$$\det(\mathbb{R}(\theta)) = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1. \tag{36}$$

5.2 3 × 3 Matrix

For 3×3 matrices, the determinant is given by the sum of $3! = 6$ terms: if $\mathbb{A}$ is a 3×3 matrix, then its determinant is

$$\det(\mathbb{A}) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}. \tag{37}$$

Fig. 7 shows two different graphical strategies to calculate this determinant. The first one, called the Sarrus' rule, is less messy: just copy the first two columns and evaluate the products of the diagonals. Again, if the diagonal comes from left to right, it is positive; if it comes from right to left, it is negative.

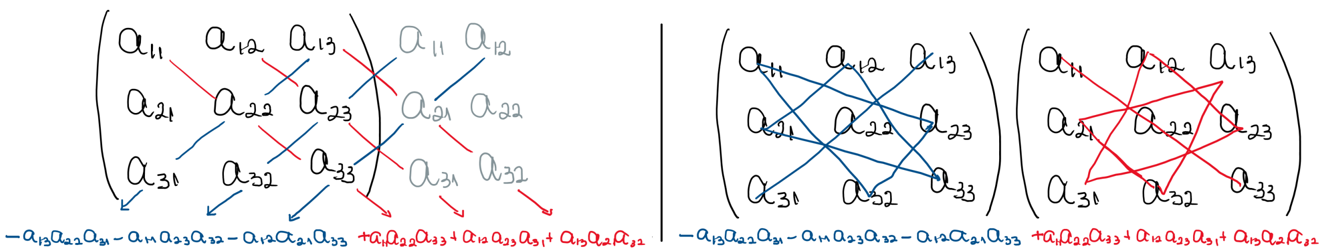

Figure 7: Caption

5.3 $n \times n$ Matrix

When the order of the matrix is higher than 3, it starts to get too cumbersome to find graphical schemes to calculate the determinant. A general way is to calculate the so called *cofactor matrix*, which essentially reduces the calculation of a determinant of order n, by at most n^2 determinants of order $n-1$. Another strategy is called "Chio's rule", which can subsequently reduce the order of a matrix. But we are not going to spend our time in such techniques.

However, for diagonal and *triangular* matrices, it is still easy. A triangular matrix is a square matrix with all its elements "below" or "above" the main diagonal equal to zero. For such matrices, the determinant is simply the product of the elements of the main diagonal,

$$\det(\mathbb{A}) = a_{11}a_{22}\dots a_{nn} \text{ , if } \mathbb{A} \text{ is a diagonal or triangular matrix.} \tag{38}$$

The reason behind it is that, for such matrices, there is only one sequence of elements satisfying "different rows and different columns" that does not pass through a zero, and this sequence is the main diagonal $\{a_{11}, a_{22}, \dots, a_{nn}\}$.

Exercise: What is the determinant of a matrix with a row (or a column) of zeros?

5.4 Properties

(i) The Identity matrix: For the identity matrix $\mathbb{I}_n$ of order n,

$$\det(\mathbb{I}_n) = 1. \tag{39}$$

(ii) It is multilinear with respect to the columns: A square matrix $\mathbb{A}$ is formed by a sequence of column vectors $\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n\end{bmatrix}$. If we linearly transform a column $\mathbf{a}_j$, i.e., $\mathbf{a}_j \to k\mathbf{a}_j + \mathbf{v}$, where k is a scalar and $\mathbf{v}$ is another column vector, then the determinant changes as

$$\det\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \dots & k\mathbf{a}_j + \mathbf{v} & \dots & \mathbf{a}_n\end{bmatrix} = k\det\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_j & \dots & \mathbf{a}_n\end{bmatrix} + \det\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{v} & \dots & \mathbf{a}_n\end{bmatrix}. \tag{40}$$

(i) It is alternating with respect to the columns: The determinant of $\mathbb{A}$ is equal to zero whenever $\mathbb{A}$ has two identical columns

$$\det\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{v} & \dots & \mathbf{v} & \dots & \mathbf{a}_n\end{bmatrix} = 0. \tag{41}$$

These three properties may be used to define the determinant: there is only one function from the space of square matrices of a given order that satisfies these three properties, and this function is called the determinant.

Other properties follow below:

(a) The transpose: The determinant of a matrix and its transpose are the same,

$$\det(\mathbb{A}^T) = \det(\mathbb{A}). \tag{42}$$

Exercise: Use this property to show that the determinant is also multilinear and alternating with respect to the rows.

(b) The product: If $\mathbb{A}$ and $\mathbb{B}$ are two square matrices of order n,

$$\det(\mathbb{A}\mathbb{B}) = \det(\mathbb{A})\det(\mathbb{B}). \tag{43}$$

Exercise: Show that a matrix $\mathbb{A}$ whose inverse satisfies $\mathbb{A}^{-1} = \mathbb{A}^T$ has determinant equal to +1 or -1.

(c) The inverse matrix: If $\mathbb{A}$ has an inverse, then

$$\det(\mathbb{A}^{-1}) = \frac{1}{\det(\mathbb{A})}. \tag{44}$$

This highlights a condition for a matrix to be invertible: A matrix $\mathbb{A}$ is invertible, i.e., it has an inverse, if, and only if, $\det(\mathbb{A}) \neq 0$.

Indeed, for the inverse of a 2×2 matrix, you can check that its determinant would appear as dividing the inverse matrix elements. (It does not prove the property, it is just a hint!)

(d) Homogeneous property: If $\mathbb{A}$ is a square matrix of order n and k is a scalar, then

$$\det(k\mathbb{A}) = k^n \det(\mathbb{A}). \tag{45}$$

Exercise: Show this result with the multilinearity property.

(e) Columns permutation: Interchanging the columns of a matrix results in changing the signal of its determinant,

$$\det\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_j & \mathbf{a}_k & \dots & \mathbf{a}_n\end{bmatrix} = -\det\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_k & \mathbf{a}_j & \dots & \mathbf{a}_n\end{bmatrix} \tag{46}$$

It is also true for the rows.

Exercise: Show this result with the multilinearity and alternating properties.

There exists many other properties of determinants, connecting them with the trace of a matrix, volumes spanned by vectors, and other mathematical quantities, but so far so good! We shall not go deeper in this subject now.

6 Linear Equations

We are going to discuss now how to solve systems of the type $\mathbb{A}\mathbf{v} = \mathbf{u}$ where $\mathbf{u}$ is a known vector and $\mathbf{v}$ is the vector one we wishes to discover. Such type of equation is very common and it is called a *linear equation*. Maybe you have already encountered it but as the form of a system of equations

$$\begin{aligned}
a_{11}v_1 + a_{12}v_2 + \ldots + a_{1m}v_m &= u_1 \\
a_{21}v_1 + a_{22}v_2 + \ldots + a_{2m}v_m &= u_2 \\
&\vdots \\
a_{n1}v_1 + a_{n2}v_2 + \ldots + a_{nm}v_m &= u_n
\end{aligned}$$

(notice that we have simply performed the matrix multiplication). The number of equations is the number of rows of $\mathbb{A}$, while the number of unknown variables v_i is the number of columns.

Let us go through an example: Suppose you want to have some cats and dogs, but you cannot spend too much money on their food and toys. You can buy (and want to spend exactly this amount) a maximum of 23 portions of food per day and 21 new toys per month. One cat eats 2 portions of food everyday, and one dog eats 3 portions of food everyday; one cat uses (destroys) 4 toys per month, while one dog uses only one per month. How many dogs and cats should you have?

Start by calling the number of cats by c and the number of dogs by d. Now we use the given information: c cats eat $2c$ portions per day and d dogs eat $3d$ portions per day, and all the portions should sum up to 23:

$$2c + 3d = 23.$$

Now, c cats use $4c$ toys per month, and d dogs use $1d$ toys per month, and all the toys should sum up to 21:

$$4c + 1d = 21.$$

These two equations combined form a system of linear equations, in which our goal is to discover the pair of numbers (c, d). In order to solve them, we can follow two different strategies.

"Solve one and substitute in the other":

This method goes like this:

1. Choose one of these equations and solve it for one of the variables, considering the other variable as a constant (e.g. find d as a function of c).
2. Use this value in the remaining equation, which is then going to be an equation for the other variable (e.g. an equation for c).
3. Solve this equation to find one of the variables (e.g. c). Use this value in the previously obtained equation in order to find the value of the remaining variable (e.g. d).

Let's do it! We can write

$$4c + d = 21 \Rightarrow d = 21 - 4c.$$

By, inserting this result in the other equation, we get

$$2c + 3d = 23 \Rightarrow 2c + 3(21 - 4c) = 23 \Leftrightarrow -10c = -40 \Leftrightarrow c = 4.$$

Now using it in the previous equation for d,

$$d = 21 - 4c \Rightarrow d = 21 - 4(4) \Leftrightarrow d = 5.$$

Therefore, you should have 4 cats and 5 dogs.

Exercise: Use this method to first find an equation for c as a function of d, which is going to give first a value of d and then for c.

"Multiply and sum":

For this method,

1. Choose one of these equations and multiply it by a well chosen number.
2. Add the obtained equation to the other equation: If you chose a good number, after summing both equations you should find an equation for only one variable.
3. Use this value to find the result for the remaining variable.

It is easier than it looks like. For instance, let us multiply the first equation by -2 and add it to the second equation:

With this method, you construct a new equation which is a linear combination of the others.

$$\begin{aligned} (-2) \times (2c + 3d = 23) \Longrightarrow\ & -4c - 6d = -46 \\ (+)\quad & \underline{\ \ 4c + 1d = 21\ \ } \\ (=)\quad & \ \ 0c - 5d = -25 \Longrightarrow d = 5. \end{aligned}$$

Now, choose any equation and substitute the value of d, for instance:

$$2c + 3d = 23 \Rightarrow 2c + 3(5) = 23 \Leftrightarrow c = 4,$$

as obtained before.

Exercise: Use the "multiply and sum" strategy to find first the value of c and then the value of d.

Inverse matrix

Notice that this system before can also be written with matrices as

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 23 \\ 21 \end{bmatrix}, \tag{47}$$

so if we multiply this equation by the inverse of $\mathbb{A} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$, we can obtain the solution:

$$\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 23 \\ 21 \end{bmatrix}. \tag{48}$$

The inverse of $\mathbb{A}$ equals to $\mathbb{A} = \frac{1}{-10} \begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix}$ and thus the solution is given by

$$\begin{bmatrix} c \\ d \end{bmatrix} = \frac{1}{-10} \begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 23 \\ 21 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \tag{49}$$

as we had before. Hence, we have seen how the solution of a linear equation may require the existence of the inverse matrix.

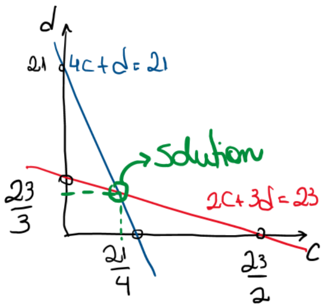

Figure 8: caption blablabla

Exercise: Solve the following linear systems:

(a) $\begin{cases} 3x + 4y = 5 \\ 1x + 4y = 3 \end{cases}$

(b) $\begin{cases} 1x + 2y + 1z = 3 \\ 4x - 2y + 1z = 3 \\ -1x + 1y - 1z = 1 \end{cases}$

6.1 Types of linear systems

The example in the previous section has a simple geometrical interpretation: Each equation describes a different line on the plane $c \times d$, and the solution is the intersection of these lines. Why is that true? Because the solution should satisfy both equations, and thus belong to both lines at the same time.

A general case for a 2-variables system is given by the three cases of Fig.9. Consider a case with 2 linear equations and 2 variables. Such system describes 2

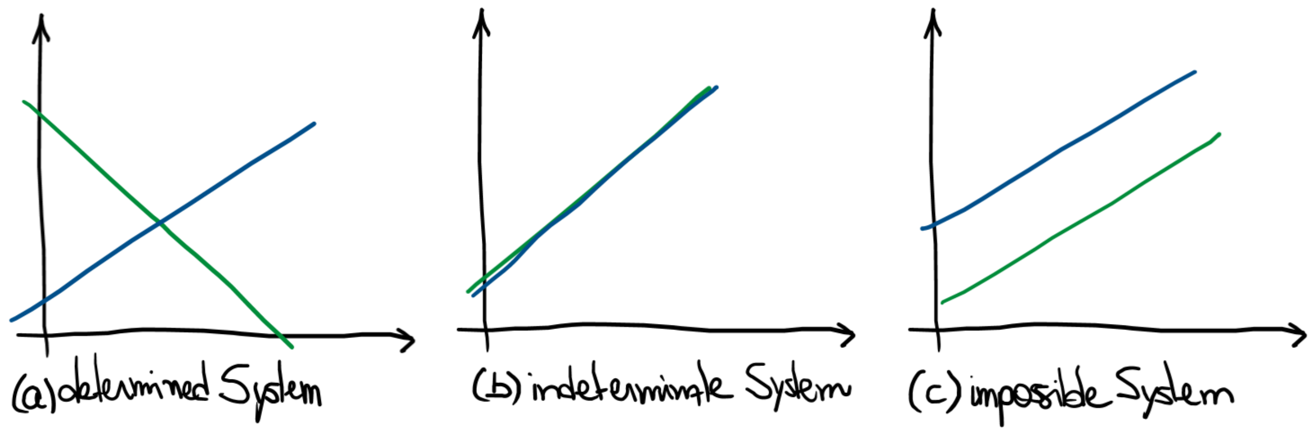

Figure 9: Caption

lines on the plane. In the first case, **(a)**, it has a unique solution and the system is called *determined*. In the second case, **(b)**, the two lines coincide, and thus the system has an infinite number of solutions. It means that one equation is a multiple of the other; such system is called *indeterminate*. In the third case, **(c)**, the lines are parallel and never touch each other. In this case, the equations have no solution and the system is called *impossible*.

This classification extents to higher dimensional systems, i.e. for matrices of order higher than two. In such a case, each equation describes a *hyperplane* and the intersection of such hyperplanes is the solution of the system. If there are less equations than the number of variables, the system is *underdetermined*: it "lacks" of some extra information to determine the exact solution. If it has more equations than variables, then it is *overdetermined*: it has more information than the necessary to determine the solution.

When the system is determined, its matrix is square and the solution can be found with its inverse.

A linear system $\mathbb{A}\mathbf{v} = \mathbf{u}$, with $\mathbb{A}$ a square matrix of order n is determined, i.e. it has a solution and it is unique, if, and only if,

$$\det(\mathbb{A}) \neq 0.$$

As already mentioned, the solution can be found if $\mathbb{A}$ has an inverse, hence its determinant should be different than zero.

7 Eigenvalues and Eigenvectors

We have done quite a good number of things so far, but let us not forget our starting point: transforming vectors. A very simple transformation is the one that does not change the vector, which is represented by the identity matrix. On the other hand, a natural question could be: are there vectors that do not change under a transformation? Mathematically, we are asking: for a given matrix $\mathbb{A}$, which vectors $\mathbf{v}$ satisfy $\mathbb{A}\mathbf{v} = \mathbf{v}$? If $\mathbb{A} = \mathbb{I}_n$, then any vector would do it, but for a different $\mathbb{A}$, the answer is not that simple.

A more general question concerns vectors that under a given transformation may change only up to a scalar. This is the concept of an *eigenvalue* problem.

Given a linear transformation represented by the square matrix $\mathbb{A}$ of order n, the *eigenvalue* problem regards finding solutions of the equation

$$\mathbb{A}\mathbf{v} = \lambda\mathbf{v}, \tag{50}$$

where λ is a scalar. The vector $\mathbf{v}$ that solves this equation is called an *eigenvector* and the scalar λ is its corresponding *eigenvalue*.

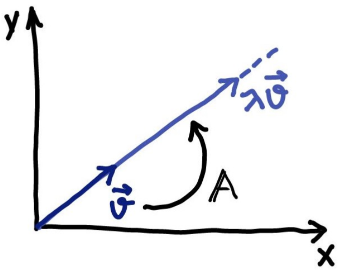

Figure 10: caption blablabla

We shall see the path to solve this system now. We start by manipulating the equation itself

$$\mathbb{A}\mathbf{v} = \lambda\mathbf{v} \Leftrightarrow \mathbb{A}\mathbf{v} - \lambda\mathbf{v} = 0 \Leftrightarrow (\mathbb{A} - \lambda\mathbb{I}_n)\mathbf{v} = \mathbf{0}.$$

Suppose the equation $\mathbb{Q}\mathbf{u} = \mathbf{0}$ where $\mathbb{Q}$ is a square matrix, $\mathbf{u}$ is a vector and $\mathbf{0}$ is the null vector. An equation like this is called *homogeneous*. However, if we aim to find this vector $\mathbf{u}$, we can multiply both sides of this equation by $\mathbb{Q}^{-1}$, if it exists: $\mathbf{u} = \mathbb{Q}^{-1}\mathbf{0}$, and then $\mathbf{u} = \mathbf{0}$. This is called the trivial solution. If we aim to find a non-trivial solution, we must assume that $\mathbb{Q}$ does not have an inverse, hence that $\det(\mathbb{Q}) = 0$.

Exercise: for which value of μ the linear system $\begin{cases} \mu x + y = 0 \\ -x + 2y = 0 \end{cases}$ has a non-trivial solution?

Therefore, to solve the eigenvalue problem, we should solve

$$\det(\mathbb{A} - \lambda\mathbb{I}_n) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} = 0, \tag{51}$$

which is a polynomial equation of order n on λ, $p_n(\lambda)$. $p_n(\lambda)$ is called the characteristic polynomial and its roots are the eigenvalues of the matrix $\mathbb{A}$. Hence, by the *fundamental theorem of algebra*, we can factorize the characteristic polynomial as

$$p_n(\lambda) = c_0(\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \dots (\lambda - \lambda_k)^{m_n},$$

where λ_i is an eigenvalue and m_i is its *multiplicity*, k is an integer smaller or equal n, and c_0 is a constant. If $m_i > 1$, then λ_i is said to be *degenerate*. Also, $m_1 + m_2 + \ldots + m_n = n$. Hence, if all eigenvalues are non-degenerate, the number of different eigenvalues of a matrix is equal to its order. However, this factorization is valid in a general case only if the eigenvalues are allowed to be complex numbers.

The first part is done: in order to calculate the eigenvalue, we first find the characteristic polynomial and find its roots. Such roots are the eigenvalues of the problem. The second step is to find the eigenvectors. And now it is easier: substitute each eigenvalue in $(\mathbb{A} - \lambda\mathbb{I}_n)\mathbf{v} = 0$ and solve the linear system. But remember that this system is indeterminate ($\det(\mathbb{A} - \lambda\mathbb{I}_n) = 0$) thus the solution is given up to a multiplicative constant. Let us apply this procedure to the matrix of the cats and dogs example and see what this all means.

1. First we find the characteristic polynomial:

$$p(\lambda) = \begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 12 = 0.$$

Solving the quadratic equation leads to the roots

$$\lambda_1 = 5 \text{ and } \lambda_2 = -2,$$

which are the eigenvalues of the matrix.

2. Insert the eigenvalues back into the equation:

$$\begin{bmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(a) For $\lambda_1 = 5$:

$$\begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} -3c + 3d \\ 4c - 4d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Notice that both lines of the resulting system give the same result: $c = d$. So supposing that d has a given value, the eigenvector of the eigenvalue λ_1 is

$$\mathbf{v}_1 = d \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

You can choose $d = 1$ for simplicity.

(b) For $\lambda_2 = -2$:

$$\begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 4c + 3d \\ 4c + 3d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and thus $c = -3d/4$. Again, suppose d is given, the eigenvector of the eigenvalue λ_2 is

$$\mathbf{v}_2 = \frac{d}{4} \begin{bmatrix} -3 \\ 4 \end{bmatrix}.$$

and you can even choose $d = 4$ for simplicity.

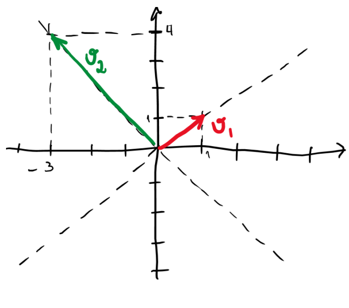

Figure 11: caption blablabla

7.1 Diagonalization

Two n matrices $\mathbb{A}$ and $\mathbb{B}$ are called *similar* if they are related through a *similarity transformation*:

$$\mathbb{B} = \mathbb{P}^{-1}\mathbb{A}\mathbb{P}, \tag{52}$$

where $\mathbb{P}$ is an invertible square matrix. A beautiful result from linear algebra is that if $\mathbb{P}$ is the matrix of eigenvectors of $\mathbb{A}$, i.e. with the eigenvectors organized as columns of $\mathbb{P}$, then $\mathbb{B}$ is a diagonal matrix with the eigenvalues of $\mathbb{A}$ in the main diagonal.

We can use this result to calculate the determinant of a matrix. First notice that 2 similar matrices have the same determinant,

$$\det(\mathbb{B}) = \det(\mathbb{P}^{-1}\mathbb{A}\mathbb{P}) = \det(\mathbb{P}\mathbb{P}^{-1}\mathbb{A}) = \det(\mathbb{A}).$$

Now, if $\mathbb{B}$ is diagonal, then

$$\det(\mathbb{A}) = \lambda_1^{m_1}\lambda_2^{m_2}\dots\lambda_k^{m_n}. \tag{53}$$

Exercise: Diagonalize $\mathbb{A} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$, i.e. write the matrix $\mathbb{P}$, $\mathbb{P}^{-1}$ and $\mathbb{D} = \mathbb{P}^{-1}\mathbb{A}\mathbb{P}$. Check that it is diagonal and that it is composed by the eigenvalues.

To diagonalize a matrix is a very useful procedure, and it is going to be very important in the next section. But let us think about its meaning:

We started asking for which vectors a linear transformation does not change their directions. We found these vectors and called them the eigenvectors of the transformation. If we consider a 2-dimensional system, if a transformation has two different eigenvectors, then it means that every vector in the plane can be constructed by a linear combination of these eigenvectors. Hence, the eigenvectors can be a basis for the plane. Therefore, if we study the system as represented on this specific basis, the effect of the transformation would be simply to stretch the coordinates of the vectors. How much would they stretch? By an amount equal to the eigenvalue associated with each eigenvector. At the end, all this information can be stored in a single procedure: a similarity transformation.

This is a very important sum up of what we have done in these notes, read and re-read it many times to be sure it is all connected in your mind and do not hesitate to ask.

8 Application: Dynamical systems

A system is dynamical if it evolves (i.e., it changes) as time passes by: cars moving, a capacitor charging and discharging, heat flowing from one place to another, chemicals passing through a membrane, a bacterium moving through a gradient, birds flocking, muscles contracting... The list of examples do not stop! And they are everywhere, from physics to biology; from chemistry to finance. And mathematics can deal with such processes in many different ways. A part of this field is devoted to *deterministic* and *continuous* systems, in which the dynamics are described as sets of *differential equations*, and this is our starting point.

A differential equation tells what is the *rate of change* of a given variable. Suppose that the size of a cat is your important variable. Thus you could ask “how does the size of the cat changes throughout its life?” The answer can be a very complicated function of many other variables, including the size itself. If you have a very big cat, maybe it is not going to grow much more; if it is very tiny, maybe it means that it is still going to grow a lot. For this example, we can write

$$\frac{dS}{dt} = f(S,\ \text{other things}\ ),$$

where S is the size of the cat and f is a continuous function. The *time derivative* $\frac{dS}{dt}$ (you can read it as "d-S-d-t") is the growth rate of the cat. In our example, the growth rate is a function of the size S itself, and of "other things".

Notice that if f is zero, it means that the cat's size does not change; if it is positive, then the cat is growing; and if it is negative, the cat's size is decreasing.

For the general case we would write

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_n; t) \text{ for } i = 1, 2, \ldots, n. \tag{54}$$

In this case, the system is composed by n different variables x_i which gives the *state* of the system, i.e. the set of values it assumes at a given moment in time. And this set is itself a vector! The rate of change of each vector's component is given by a continuous function f_i, which possibly depends on all the variables in the system, and even on time itself. When the dynamical system does not depend *explicitly* on time, it is called *autonomous*, and we shall stick to this case. In a vector notation, such system can be written as

Notation: The time derivative may also be represented by a dot over the variable, like $\frac{dx}{dt} \equiv \dot{x}$.

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}), \tag{55}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$; $\frac{d\mathbf{x}}{dt} = (\frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt})$ and $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$.

8.1 Linear systems

Let us start with f being a linear transformation, i.e., it can be expressed as a matrix,

$$\frac{d\mathbf{x}}{dt} = \mathbb{A}\mathbf{x}. \tag{56}$$

Our objective is to find the vector $\mathbf{x}$ as a function of time.

For those versed in the mathematics of differential equations, it may be a almost direct to solve it as

$$\mathbf{x}(t) = \mathbf{x}(t_0)e^{\mathbb{A}(t-t_0)}, \tag{57}$$

but do not forget that $\mathbb{A}$ is a matrix, not a real number. However, let us use a different method here.

Let Λ be the matrix that diagonalizes $\mathbb{A}$, i.e. $\Lambda^{-1}\mathbb{A}\Lambda = \mathbb{D}$, where $\mathbb{D}$ is the diagonal matrix of $\mathbb{A}$. Hence,

$$\Lambda^{-1}\frac{d\mathbf{x}}{dt} = \Lambda^{-1}\mathbb{A}\Lambda\Lambda^{-1}\mathbf{x},$$

where we have multiplied both sides by Λ^{-1} and inserted an identity matrix $\mathbb{I}_n = \Lambda^{-1}\Lambda$ in the middle of the right side. But the derivative is a commutative linear operator, and then

$$\frac{d(\Lambda^{-1}\mathbf{x})}{dt} = (\Lambda^{-1}\mathbb{A}\Lambda)(\Lambda^{-1}\mathbf{x}) = \mathbb{D}(\Lambda^{-1}\mathbf{x}).$$

Define the new vector $\mathbf{y} \equiv \Lambda^{-1}\mathbf{x}$ and we finally find

$$\frac{d\mathbf{y}}{dt} = \mathbb{D}\mathbf{y},$$

which I know it does not seem different from where we started, but remember that $\mathbb{D}$ is diagonal, and because of that every variable y_i is *decoupled* from all the others,

$$\frac{dy_i}{dt} = \lambda_i y_i, \tag{58}$$

where λ_i is the i-th eigenvalue of $\mathbb{A}$. The solution of this equation is the exponential $y_i(t) = y_i(t_0)e^{\lambda_i(t-t_0)}$, and as it can be seen, the eigenvalues play a very important role here: supposing λ_i is a real number, if $\lambda_i < 0$ then y_i decreases to zero as time passes; but if $\lambda_i > 0$, then y_i increases exponentially as time passes. Moreover, if λ is a complex number, then the solution may oscillate.

However, we were interested in how $\mathbf{x}$ behaves, which we can write as

$$\mathbf{x} = \Lambda\mathbf{y}$$

and remember that the matrix Λ is composed by the eigenvectors of $\mathbb{A}$ in its columns. Let the eigenvectors be called by $\mathbf{v}_i$, then

$$\mathbf{x} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{bmatrix}\begin{bmatrix} y_1(t_0)e^{\lambda_1(t-t_0)} \\ y_2(t_0)e^{\lambda_2(t-t_0)} \\ \vdots \\ y_n(t_0)e^{\lambda_n(t-t_0)}\end{bmatrix} = \mathbf{v}_1 y_1(t_0)e^{\lambda_1(t-t_0)} + \mathbf{v}_2 y_2(t_0)e^{\lambda_2(t-t_0)} + \ldots + \mathbf{v}_n y_n(t_0)e^{\lambda_n(t-t_0)}. \tag{59}$$

This is a beautiful result! It says that the dynamics of the system is a linear combination of the dynamics of its eigenvectors,* which is an easier dynamics to describe. Furthermore, this linear combination is defined by the initial conditions of the system.

*I mean: the dynamics in the direction of the eigenvectors, which can be called the *eigendirections*.

Suppose that, for instance, all the eigenvalues have negative real part. It means that all the exponential factors in Eq. (59) decreases to zero as the time passes, therefore, in the long run (for very large time values) $\mathbf{x}$ is going to approach zero, regardless of the initial conditions. This is a very nice model for a stable system. Suppose that $\mathbf{x}$ is a perturbation on a given system. If such system is not supposed to leave the state it is, you want the oscillations to vanish away, regardless of how you perturb it at the beginning. Think about your skin after someone pinches you: it returns to its "unpinched" state after a few seconds.

But suppose now that there is one eigenvalue whose real part is larger than the others, say λ_1. Thus, in the long run, the exponential $e^{\mathcal{R}e(\lambda_1)}t$ is a much larger number than all the other exponential factors. Therefore, after a *transient period* the system is going to be fairly described by the dynamics of the eigenvector $\mathbf{v}_1$.

An example

Consider the following system

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}2 & 3 \\ 4 & 1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix},$$

where we chose the same matrix $\mathbb{A}$ as in the cats-dogs example of the previous section. Setting $t_0 = 0$, we can use Eq.(59) to find the solution of this equation,

$$\mathbf{x}(t) = y_1(0)\begin{bmatrix}1 \\ 1\end{bmatrix}e^{5t} + y_2(0)\begin{bmatrix}-3 \\ 4\end{bmatrix}e^{-2t},$$

and the only thing that remains to be found are the initial conditions written as y: if at time $t = 0$, $x_1(0) = x_{1,0}$ and $x_2(0) = x_{2,0}$, with the relation $\mathbf{y} = \Lambda^{-1}\mathbf{x}$, we can find

$$\mathbf{x}(t) = (4x_{1,0} + 3x_{2,0}) \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + (-x_{1,0} + x_{2,0}) \begin{bmatrix} -3 \\ 4 \end{bmatrix} e^{-2t}.$$

In the long run (for $t \gg 1$),

$$\mathbf{x}(t) \sim \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t}.$$

This is called the *asymptotic behaviour* of the system.

8.2 Non-linear dynamics

If the function f that defines the dynamical system is non-linear, i.e., it fails in satisfying $f(\mathbf{x} + \lambda\mathbf{y}) = f(\mathbf{x}) + \lambda f(\mathbf{y})$ for every vector $\mathbf{x}$ and $\mathbf{y}$ and every scalar λ, then the dynamics is called non-linear. Examples of non-linear functions are everywhere, $x^2, x^3, \ldots, \cos x, \sin x, \ldots, \log x$ and e^x are all non-linear functions, and it is a great issue that for the great majority of non-linear systems, we are not able to find a solution. So what can be done?

To such systems, we try to study their behaviours without solving them. The first step is to find where it stops changing:

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}) = 0. \tag{60}$$

The points $\mathbf{x}^*$ for which the time-derivative is null are called *fixed points* of the system. If the system starts exactly on a fixed point, it remains there forever. But what if it starts very close to a fixed point: is it going to move towards it or is it going to drift away from it? These questions concern the concept of *stability*.

Exercise: Find the fixed points of the *Logistic growth equation*

$$\dot{x} = rx\left(1 - \frac{x}{K}\right).$$

This equation describes the growth of a homogeneous population in a resource-limited environment. Based on the sign that the growth rate assumes (positive or negative), argue that the system always reaches $x = k$ whenever it starts with a positive size population (i.e. $x = k$ is a stable fixed point).

8.3 Linear stability analysis

In order to study the stability of a fixed point, it is a valid approach to study the behaviour of the system in a small neighborhood of this point. When we zoom into a point, locally its dynamics looks linear and then we can use the theory developed so far. Let us see how to do it.

The trick here is to use a Taylor series to expand the function f around a fixed point $\mathbf{x}^*$. Keeping only first-order terms, (i.e. x_i^0 and x_i^1),

$$\frac{d\mathbf{x}}{dt} \approx f(\mathbf{x}^*) + \mathbb{J}_f(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \tag{61}$$

where $\mathbb{J}_f(\mathbf{x}^*)$ is the *Jacobian* matrix of f calculated at the point $\mathbf{x} = \mathbf{x}^*$. Such matrix is defined as

$$\mathbb{J}_f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}, \tag{62}$$

or in terms of its elements,

$$J_{ij} = \frac{\partial f_i}{\partial x_j}, \tag{63}$$

and it plays a central role here.

Now, because $\mathbf{x}^*$ is a fixed point, $f(\mathbf{x}^*) = 0$ and because it is a constant vector, $\frac{d\mathbf{x}^*}{dt} = 0$. Then, if we define $z \equiv \mathbf{x} - \mathbf{x}^*$, we find

$$\frac{d(\mathbf{x} - \mathbf{x}^*)}{dt} \approx \mathbb{J}_f(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \Rightarrow \frac{d\mathbf{z}}{dt} \approx \mathbb{J}_f(\mathbf{x}^*)\mathbf{z}.$$

Notice that the variable $\mathbf{z}$ as defined above represents a small deviation from the fixed point $\mathbf{x}^*$ and when it is very small ($\|\mathbf{z}\| \ll 1$), it is well described by a linear equation. Thus, the eigenvalues and eigenvectors of the Jacobian matrix calculated on a fixed point is going to determine if the components of $\mathbf{z}$ increases or decreases as time passes. If at least one component increases, then it is an *unstable* fixed point, but if all components decrease, then the fixed point is called *stable*.

Such information (provided by the Jacobian matrix) is summarized in the following box:

Let $\mathbf{x}^*$ be the fixed point of a the dynamical system $\dot{\mathbf{x}} = f(\mathbf{x})$, i.e., $f(\mathbf{x}^*) = 0$. Consider λ_1 to be the eigenvalue with the largest real part. Then, if $\mathrm{Re}(\lambda_1) < 0$, then the fixed point is stable. If at least one eigenvalue of the Jacobian has positive real part, then the fixed point is unstable.